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Proposed Method for Expressing Inner Derivations of Leibniz Algebras

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Abstract

This work concentrates on the aspect of computation of a specific type of derivation mapping referred to as 'inner derivation', which is essential for the algebraic systems' structure. In this context, we provide an algebraic method to determine inner derivations of Leibniz algebras within finite dimensions expressed as matrices. Our method is inherent in the crucial properties of Leibniz algebras, allowing for the specified construction and determination of inner derivations effectively throughout different dimensions. To demonstrate how invaluable and efficient the proposed method is, we apply it on four-dimensional complex algebras of Leibniz that have been completely classified in the literature. For each of the algebras considered, we put forward comprehensive descriptions of the corresponding inner derivations, demonstrating the sequential computational approach and highlighting the specific algebraic properties that influence the arrangement of these derivations. We focus on inner derivations, but we also look more closely at the derivation space by building and showing the span basis for the outer derivations that go with each algebra. This dual analysis makes it clear how inner and outer derivations give a full picture of the derivation structure for these algebras. The results indicate that our computational method serves as an effective instrument for advancing the investigation of derivations in Leibniz algebras with limited dimensions and may serve as a foundation for additional research in analogous algebraic contexts.

Keywords: Inner derivation, Leibniz algebra, and nilpotence.

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1. Introduction

As a generalisation of Lie algebras (Bloh, 1965) presented Leibniz algebras in 1965, and (Loday, 1993) conducted an extensive study of them in 1993. Since that time, Leibniz algebras have gained importance due to their applications in homology, non-commutative geometry, and deformation theory and are extensively studied by the group of investigators such as (Mohammed et al., 2022, Mondal & Saha, 2023, Edalatzadeh et al., 2024, Bhutia & Behera, 2025, Fehlberg Junior et al., 2025, and Li & Wang, 2025).

One of the essential thoughts within the studies of any algebra is derivations, inner and outer derivations; see, for example, (Abdurasulov et al., 2022, Shermatova & Khudoyberdiyev, 2023, Wu et al., 2023, Kurdachenko et al., 2024, Ansari, 2025, and Bell & Buzaglo, 2025). All of these derivations are an invaluable tool for understanding the structural characteristics of algebras. In particular, their dimensions are the critical invariant in considering the geometric classification of any algebra stated in the literature (Mohammed et al., 2017, Benayadi & Mhamdi, 2022, Mancini, 2023, Guo & Das, 2024, Abdelwahab et al., 2025, and Mondal & Saha, 2025).

Moreover, (Bermúdez & Campoamor-Stursberg, 2013) presented the notion of inner derivations. It is a right multiplication operator of a Lie algebra derivation. The derivations of outer are construed as elements of the first cohomology space that have essential techniques of algebraic topology; see (Arutyunov & Naianzin, 2022, Rubio y Degrassi et al., 2023, Peligrad, 2024, Manju & Sharma, 2025, and Mohammed et al., 2025). In this case, the inner derivations constitute 1-coboundaries, and the derivations are 1-cocycles. The motivation for conducting this study was to enhance our comprehension of the structure and symmetries inherent in algebraic systems, specifically in Lie algebras and ring theory. This distinction can ascertain which symmetries (automorphisms) of the algebra are intrinsic

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and which are derived from external structures. There are also numerous applications for them in math, physics, cryptography, and coding theory.

There are numerous theoretical investigations on these derivations, and so on. However, their computation is still an open problem. Consequently, the reason we are interested in this study is to propose a method for representing inner derivations expressed as matrices for any finite-dimensional Leibniz algebra. Our applications will be restricted to dimension four Leibniz algebras with nilpotence. In addition, we use our results and the previously acquired derivation result to decide the span basis of outer derivatives.

2. Building Foundations

Let us provide some significant concepts that are all essential for the understanding of this work.

Definition 2.1: Suppose L represents a linear space over the field K is referred to as an algebra of Leibniz whenever it achieves its identity:

$$[\mathfrak{H},[\wp,\mathfrak{I}]] = [[\mathfrak{H},\wp],\mathfrak{I}] - [[\mathfrak{H},\mathfrak{I}],\wp]$$

$\forall\, \mathfrak{H},\wp,\mathfrak{I} \in L$, whereas $[\cdot,\cdot]$ indicates the multiplication of L.

Definition 2.2. Suppose L represents algebra of Leibniz. Then, it is stated nilpotent whenever $L^{\,n} = \{0\}$.

Definition 2.3. Suppose $\mathfrak{D}: L \to L$ refers to a map of derivation of L whenever it is stated as below:

$$\mathfrak{D}([\mathfrak{H},\wp]) = [\mathfrak{D}(\mathfrak{H}),\wp] + [\mathfrak{H},\mathfrak{D}(\wp)] \qquad \forall\, \mathfrak{H},\wp \in L.$$

Definition 2.4. Suppose L represents Leibniz algebra and $\wp$ denotes a vector in L. The multiplication of right $\mathcal{R}_{\wp}: L \to L$ is considered derivation of inner and it is defined below:

$$\mathcal{R}_{\wp}(\mathcal{U}) = [\mathfrak{H},\mathcal{V}] \qquad \forall\, \mathfrak{H} \in L.$$

The collection of each L derivations (resp. inner derivations) is indicated to $Der(L)$(resp. $Inn(L)$).

The quotient space $Out(L,L) = Der(L)/Inn(L)$ is referred to as a space of outer derivatives. It can be considered as the Leibniz's one-dimensional Hochschild cohomology. In other words $H^1(L,L) \approx Out(L,L)$.

In this case study, it is assumed that every algebra is specified over the complex number field.

3. The Proposed Method for Finding the Inner Derivations and Its Applications

In this section, we explain our proposed method and its applications to describe the inner derivations for Leibniz algebras. The effectiveness of the proposed method lies in its ability to solve a system of linear algebraic equations that depend solely on structure constants and on the dimension of Leibniz algebras. We can obtain the structure constants through Leibniz's algebraic classification. The system is made up of n^2 linear equations, which are expressed in $(n \times n)$ variables. Here, n represents the dimension of the Leibniz algebras. Solving this system of linear algebraic equations produces a result in matrix form, with dimensions of $n \times n$.

The representation in a matrix for the derivations of inner provides many significant benefits in Leibniz algebras for any dimension. To begin with, it offers a cohesive and structured procedure, enabling the computation and classification of derivations due to established methodologies of linear algebra. This matrix-based technique

facilitates computations, and is applicable in both theoretical studies and practical algorithmic applications within contemporary algebraic research. Furthermore, by enabling efficient algorithmic execution, such as that found in Maple software, it becomes scalable for algebras of larger dimensions. Algorithms can systematically and transparently address derivations via the matrix format, thereby enabling the methodical analysis of linear relationships and the resolution of associated equations.

Next, we roughly characterize the theoretical basis of the method.

Suppose L is Leibniz algebra for n-dimension. By providing the basis $\{\mathcal{E}_1,\mathcal{E}_2,\cdots,\mathcal{E}_{\mathrm{n}}\}$ in L, elements $\mathfrak{H}$ and $\wp$ of L are able to expressed below:

$$\mathfrak{H} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathcal{u}^{\mathrm{i}}\, \mathcal{E}_{\mathrm{i}} \qquad \text{and} \qquad \wp = \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathcal{v}^{\mathrm{j}}\, \mathcal{E}_{\mathrm{j}}$$

The products $[\mathcal{U},\mathcal{V}]$ is totally determined by the products $[\mathcal{E}_i, \mathcal{E}_j]$, which is:

$$[\mathcal{E}_i, \mathcal{E}_j] = \sum_{k=1}^{n} \gamma_{ij}^{k}\, \mathcal{E}_k, \qquad \forall i, j \in N$$

Where γ_{ij}^{k} are known as the structure constants of L.

To find the inner derivations of L, consider $\mathfrak{J} = \mathcal{W}_1\mathcal{E}_1 + \mathcal{W}_2\mathcal{E}_2 + \cdots + \mathcal{W}_n\mathcal{E}_n$ as a fixed vector in L. Let $R_{\mathfrak{J}}$ be arise a derivation of inner. More particularly,

$$\mathcal{R}_{\mathfrak{J}}(\mathcal{E}_i) = [\, \mathcal{E}_i, \mathfrak{J}\,] \tag{1}$$

The linear transformation $R_{\mathcal{W}}$ can be expressed in a matrix form

$$\mathcal{R}_{\mathfrak{J}}(\mathcal{E}_i) = \textstyle\sum_{j=1}^{n} d_{ji}\, \mathcal{E}_j \tag{2}$$

Definitely, that leads to

$$\begin{aligned} \textstyle\sum_{j=1}^{n} d_{ji}\, \mathcal{E}_j &= [\, \mathcal{E}_i, \mathfrak{J}\,] \\ &= \mathcal{E}_i \textstyle\sum_{t=1}^{n} \mathcal{W}_t\, \mathcal{E}_t \\ &= \textstyle\sum_{t=1}^{n} \mathcal{W}_t \sum_{j=1}^{n} \gamma_{it}^{j}\, \mathcal{E}_j \end{aligned} \tag{3}$$

Thus, we obtain

$$\textstyle\sum_{j=1}^{n} d_{ji}\, \mathcal{E}_j = \sum_{j=1}^{n} \sum_{t=1}^{n} \mathcal{W}_t\, \gamma_{it}^{j}\, \mathcal{E}_j \tag{4}$$

Equated the coefficient in the basis vectors resulted in the subsequent system of equations:

$$d_{ji} = \textstyle\sum_{t=1}^{n} \mathcal{W}_t \gamma_{it}^{j} \tag{5}$$

The final equations, together with the structure constants of L, are solved to obtain the linear transformation $\mathcal{R}_{\mathcal{W}}$ in the following matrix representation.

$$\mathcal{R}_{\mathcal{W}} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix}$$

To show the effectiveness and easily implemented of the purposed method, we apply it to describe the derivation of the inner for Leibniz algebras in four dimensions. In this case the system (4) has the following form:

$$\begin{aligned}
d_{11} &= \mathcal{W}_1\gamma_{11}^1 + \mathcal{W}_2\gamma_{12}^1 + \mathcal{W}_3\gamma_{13}^1 + \mathcal{W}_4\gamma_{14}^1 \\
d_{12} &= \mathcal{W}_1\gamma_{21}^1 + \mathcal{W}_2\gamma_{22}^1 + \mathcal{W}_3\gamma_{23}^1 + \mathcal{W}_4\gamma_{24}^1 \\
d_{13} &= \mathcal{W}_1\gamma_{31}^1 + \mathcal{W}_2\gamma_{32}^1 + \mathcal{W}_3\gamma_{33}^1 + \mathcal{W}\gamma_{34}^1 \\
d_{14} &= \mathcal{W}_1\gamma_{41}^1 + \mathcal{W}_2\gamma_{42}^1 + \mathcal{W}_3\gamma_{43}^1 + \mathcal{W}_4\gamma_{44}^1 \\
d_{21} &= \mathcal{W}_1\gamma_{11}^2 + \mathcal{W}_2\gamma_{12}^2 + \mathcal{W}_3\gamma_{13}^2 + \mathcal{W}_4\gamma_{14}^2 \\
d_{22} &= \mathcal{W}_1\gamma_{21}^2 + \mathcal{W}_2\gamma_{22}^2 + \mathcal{W}_3\gamma_{23}^2 + \mathcal{W}_4\gamma_{24}^2 \\
d_{23} &= \mathcal{W}_1\gamma_{31}^2 + \mathcal{W}_2\gamma_{32}^2 + \mathcal{W}_3\gamma_{33}^2 + \mathcal{W}_4\gamma_{34}^2 \\
d_{24} &= \mathcal{W}_1\gamma_{41}^2 + \mathcal{W}_2\gamma_{42}^2 + \mathcal{W}_3\gamma_{43}^2 + \mathcal{W}_4\gamma_{44}^2 \\
d_{31} &= \mathcal{W}_1\gamma_{11}^3 + \mathcal{W}_2\gamma_{12}^3 + \mathcal{W}_3\gamma_{13}^3 + \mathcal{W}_4\gamma_{14}^3 \\
d_{32} &= \mathcal{W}_1\gamma_{21}^3 + \mathcal{W}_2\gamma_{22}^3 + \mathcal{W}_3\gamma_{23}^3 + \mathcal{W}_4\gamma_{24}^3 \\
d_{33} &= \mathcal{W}_1\gamma_{31}^3 + \mathcal{W}_2\gamma_{32}^3 + \mathcal{W}_3\gamma_{33}^3 + \mathcal{W}_4\gamma_{34}^3 \\
d_{34} &= \mathcal{W}_1\gamma_{41}^3 + \mathcal{W}_2\gamma_{42}^3 + \mathcal{W}_3\gamma_{43}^3 + \mathcal{W}_4\gamma_{44}^3 \\
d_{41} &= \mathcal{W}_1\gamma_{11}^4 + \mathcal{W}_2\gamma_{12}^4 + \mathcal{W}_3\gamma_{13}^4 + \mathcal{W}_4\gamma_{14}^4
\end{aligned} \tag{6}$$

$$d_{42} = \mathcal{W}_1\gamma_{21}^4 + \mathcal{W}_2\gamma_{22}^4 + \mathcal{W}_3\gamma_{23}^4 + \mathcal{W}_4\gamma_{24}^4$$
$$d_{43} = \mathcal{W}_1\gamma_{31}^4 + \mathcal{W}_2\gamma_{32}^4 + \mathcal{W}_3\gamma_{33}^4 + \mathcal{W}_4\gamma_{34}^4$$
$$d_{44} = \mathcal{W}_1\gamma_{41}^4 + \mathcal{W}_2\gamma_{42}^4 + \mathcal{W}_3\gamma_{43}^4 + \mathcal{W}_4\gamma_{44}^4$$

Let d_{ij} be elements of inner derivation matrix as follows:

$$\mathcal{R}_{\mathfrak{J}} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix}$$

By using the algebraic classification that was obtained from (Ayupov et al., 2019), we get this result.

Theorem 3.1. The derivations of inner for dimension four Leibniz algebras with nilpotence are illustrated subsequently:

Table 1. The inner derivations for dimension four Leibniz algebras with nilpotence.

IC	Inner derivation	Dim($R_{\mathfrak{J}}$ (L))
L_4^1	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathcal{W}_1 & 0 & 0 & 0 \\ 0 & \mathcal{W}_1 & 0 & 0 \\ 0 & 0 & \mathcal{W}_1 & 0 \end{bmatrix}$	1
L_4^2	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_1 & \mathcal{W}_1 & 0 & 0 \\ \mathcal{W}_2 & 0 & \mathcal{W}_1 & 0 \end{bmatrix}$	2
L_4^3	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_1 & \mathcal{W}_1 & 0 & 0 \\ 0 & 0 & \mathcal{W}_1 & 0 \end{bmatrix}$	1
L_4^4		2
L_4^5	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_1 & \mathcal{W}_1 & 0 & 0 \\ \alpha\mathcal{W}_2 & \mathcal{W}_2 & \mathcal{W}_1 & 0 \end{bmatrix}$	2
L_4^6	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_1 & 0 & 0 & 0 \\ \mathcal{W}_2 & 0 & \mathcal{W}_1 & 0 \end{bmatrix}$	2
L_4^7	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_1 & 0 & 0 & 0 \\ 0 & \mathcal{W}_2 & \mathcal{W}_1 & 0 \end{bmatrix}$	3
L_4^8	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathcal{W}_2 & \mathcal{W}_1 & 0 & 0 \\ \mathcal{W}_1 - \mathcal{W}_3 & 0 & \mathcal{W}_1 & 0 \end{bmatrix}$	3
L_4^9		3
L_4^{10}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathcal{W}_2 & \mathcal{W}_1 & 0 & 0 \\ \mathcal{W}_1 + \mathcal{W}_2 - \mathcal{W}_3 & 0 & \mathcal{W}_1 & 0 \end{bmatrix}$	

IC	Inner derivation	Dim($R_{\mathfrak{J}}$ (L))
		3
L_4^{11}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathcal{W}_2 & \mathcal{W}_1 & 0 & 0 \\ \mathcal{W}_1+2\mathcal{W}_2-\mathcal{W}_3 & \mathcal{W}_2 & \mathcal{W}_1 & 0 \end{bmatrix}$	3
L_4^{12}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathcal{W}_2 & \mathcal{W}_1 & 0 & 0 \\ \mathcal{W}_1-\mathcal{W}_3 & \mathcal{W}_2 & \mathcal{W}_1 & 0 \end{bmatrix}$	2
L_4^{13}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_2 & -\mathcal{W}_1-2\mathcal{W}_2 & 0 & 0 \\ \mathcal{W}_1 & \mathcal{W}_2 & 0 & 0 \end{bmatrix}$	2
L_4^{14}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_2 & -\mathcal{W}_2 & 0 & 0 \\ 0 & \mathcal{W}_1 & 0 & 0 \end{bmatrix}$	2
L_4^{15}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_1 & -\alpha\mathcal{W}_1 & 0 & 0 \\ \mathcal{W}_2 & -\mathcal{W}_2 & 0 & 0 \end{bmatrix}$	3
L_4^{16}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_1+\alpha\mathcal{W}_2 & -\alpha\mathcal{W}_1+\mathcal{W}_2 & 0 & 0 \end{bmatrix}$	3
L_4^{17}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_2+\mathcal{W}_3 & -\mathcal{W}_1+\mathcal{W}_2 & \mathcal{W}_1 & 0 \end{bmatrix}$	2
L_4^{18}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_1+\mathcal{W}_2 & -\mathcal{W}_1 & \mathcal{W}_3 & 0 \end{bmatrix}$	2
L_4^{19}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_2 & 0 & 0 & 0 \\ 0 & \mathcal{W}_1 & 0 & 0 \end{bmatrix}$	2
L_4^{20}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_2 & -\mathcal{W}_1 & 0 & 0 \\ 0 & \mathcal{W}_2 & 0 & 0 \end{bmatrix}$	2
L_4^{21}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathcal{W}_2 & 0 & 0 \\ 0 & \mathcal{W}_1 & 0 & 0 \end{bmatrix}$	3

IC	Inner derivation	Dim($R_{\mathfrak{J}}$ (L))
	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathcal{W}_2 & 0 & 0 \\ \mathcal{W}_2 & \frac{1+\alpha}{1-\alpha}\mathcal{W}_1 & 0 & 0 \end{bmatrix}$	
	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{W}_2 & -\mathcal{W}_1 & \mathcal{W}_3 & 0 \end{bmatrix}$	

Proof**.** Let us consider L_4^1: $\mathcal{E}_1.\mathcal{E}_1 = \mathcal{E}_2, \mathcal{E}_2.\mathcal{E}_1 = \mathcal{E}_3$, and $\mathcal{E}_3.\mathcal{E}_1 = \mathcal{E}_4$. Then, we have $\gamma_{11}^2 = 1, \gamma_{21}^3 = 1, \gamma_{31}^4 = 1$ and all the rest are zeros. Applying the system of equations (6), we have $d_{21} = \mathcal{W}_1,\ d_{32} = \mathcal{W}_1,\ d_{43} = \mathcal{W}_1$, and the others are zeros. Hence,

$$\mathcal{R}_{\mathfrak{J}} = \mathrm{d_{ij}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathcal{W}_1 & 0 & 0 & 0 \\ 0 & \mathcal{W}_1 & 0 & 0 \\ 0 & 0 & \mathcal{W}_1 & 0 \end{bmatrix}$$

The remaining cases may be handled similarly with a slight adjustment.

Using the results obtained from (Abdou & Mosbahi, 2025) and Theorem 3.1, we are able to obtain span bases of outer derivations as stated in the next corollary.

Corollary 3.1: The derivations of outer space for 4-dimensional Leibniz algebras can be found below:

$Out(L_4^1, L_4^1) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{31}}, \overline{E_{41}}\}$;
$Out(L_4^2, L_4^2) = span_{\mathbb{C}}\{\overline{E_{21}}, \overline{E_{42}}\}$;
$Out(L_4^3, L_4^3) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}, \overline{E_{41}}, \overline{E_{43}}\}$;
$Out(L_4^4, L_4^4) = span_{\mathbb{C}}\{\overline{E_{42}}\}$;
$Out(L_4^5, L_4^5) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}, \overline{E_{42}}\}$;
$Out(L_4^6, L_4^6) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{41}}\}$;
$Out(L_4^7, L_4^7) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}\}$;
$Out(L_4^8, L_4^8) = span_{\mathbb{C}}\{\overline{E_{21}}, \overline{E_{42}}\}$;
$Out(L_4^9, L_4^9) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{43}}\}$;
$Out(L_4^{10}, L_4^{10}) = span_{\mathbb{C}}\{\overline{E_{42}}\}$;
$Out(L_4^{11}, L_4^{11}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{32}}, \overline{E_{42}}\}$;
$Out(L_4^{12}, L_4^{12}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{32}}, \overline{E_{41}}\}$;
$Out(L_4^{13}, L_4^{13}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{32}}, \overline{E_{42}}\}$;
$Out(L_4^{14}, L_4^{14}) = span_{\mathbb{C}}\{\overline{E_{12}}, \overline{E_{44}}\}$;
$Out(L_4^{15}, L_4^{15}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}\}$;
$Out(L_4^{16}, L_4^{16}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}\}$;
$Out(L_4^{17}, L_4^{17}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{22}}, \overline{E_{32}}, \overline{E_{42}}\}$;
$Out(L_4^{18}, L_4^{18}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{12}}, \overline{E_{22}}, \overline{E_{41}}, \overline{E_{42}}\}$;
$Out(L_4^{19}, L_4^{19}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{12}}, \overline{E_{22}}, \overline{E_{31}}, \overline{E_{41}}\}$;
$Out(L_4^{20}, L_4^{20}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{12}}, \overline{E_{22}}, \overline{E_{31}}, \overline{E_{41}}\}$;
$Out(L_4^{21}, L_4^{21}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{12}}, \overline{E_{21}}, \overline{E_{22}}\}$.

4. Conclusion

This study provides the best way to explain the derivations of inner for Leibniz algebras with any finite dimension. We applied this method to four-dimensional nilpotent complex algebras of Leibniz to demonstrate its efficacy and ease of use. We then found the span bases for all of the outer derivations. The dimensions on the inner derivations are from zero to nine. The dimensions of outer derivations can also be between one and five. These dimensional applications are important constants when classifying algebras geometrically and can be used in many different areas. In this study, the method facilitates the explanation of derivation structures in algebras of Leibniz, that is critical for comprehending their properties. By clearly computing the derivations of the inner for specific classes of algebra, we present the framework for subsequent research into the symmetries and expansion properties of the mathematical objects. Smaller classified groups of nilpotent algebras of Leibniz are carried out and simplified through the new invariants. This is beneficial for both the mathematical theory and applications in practice, where algebraic structures have a critical role, for instance, in theoretical physics. Prospective research may investigate how the proposed techniques can be implemented in algebras of Leibniz with higher dimensions as well as how the cohomology spaces and derivation are connected. This could help us learn more about these algebras and how to classify them. In conclusion, the results obtained underscore the utility and versatility of the proposed method, highlighting its potential as a beneficial tool in the ongoing study of algebraic structures and their properties.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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